MANIFOLDS OF N DIMENSIONS.*

By O. Veblen and J. W. Alexander, II.

Introduction.

1. A complete classification of manifolds from the point of view of analysis situs still remains to be made, althought Betti† and Riemann‡ have shown that with every n-dimensional manifold there may be associated a set of constants

$$B_1, \quad B_2, \quad B_3, \quad \cdots, \quad B_{n-1}$$

which are obtained by generalizing the notion of the connectivity of a surface.

Poincaré has proved§ that any manifold M_n may be completely characterized from a topological point of view by means of certain suitably chosen matrices, and has shown how to derive from these matrices a set of positive integers||

$$P_1, \quad P_2, \quad P_3, \quad \cdots, \quad P_{n-1}$$

which are invariants of M_n. These numbers have been called by him the "Betti numbers" on account of to their close resemblance to the numbers B_i of Betti and Riemann. Poincaré has also used the matrices in deriving his *coefficients of torsion* as well as in discussing the *fundamental group* of a manifold. The numbers P_i determined by a given *two-sided* manifold satisfy two relations, a theorem of duality

$$(1) \qquad P_i = P_{n-i}$$

and the generalized Euler theorem

$$(2) \qquad \sum_0^n (-1)^i \alpha_i = 1 + (-1)^n + \sum_1^{n-1} (-1)^i (P_i - 1),$$

where the α's have the meaning defined in § 4 below. The numbers P_i associated with a one-sided manifold satisfy the relation

* Read before American Mathematical Society, February 22, 1913.

† Annali di Matematica (2), vol. 4 (1871), p. 140.

‡ Werke, 2d edition, p. 474.

§ Journal de l'Ecole Polytechnique, vol. 1 (1895), p. 1; Rendiconti del Circolo Matematico di Palermo, vol. 13 (1899), p. 285; Proceedings of London Math. Society, vol. 32 (1900), p. 277.

|| A definition of these constants is given in § 14 below together with a proof of formulas (2) and (3).

$$\sum_{0}^{n}(-1)^i\alpha_i = 1 + \sum_{1}^{n-1}(-1)^i(P_i - 1) \tag{3}$$

and do not satisfy the duality theorem.*

2. In this paper, we attempt to establish some of the fundamental definitions and theorems as simply and rigorously as possible, so as to furnish an introduction to the memoirs of Poincaré. At the same time, we propose to show that a manifold may be described by means of certain systems of linear equations reduced modulo 2.* These equations lead us to a set of space constants

$$R_1, \quad R_2, \quad R_3, \quad \cdots, \quad R_{n-1}$$

which closely resemble the numbers P of Poincaré, both with respect to the geometric interpretation which may be given to them and in that they satisfy a duality theorem

$$R_i = R_{n-i} \tag{4}$$

and a generalized Euler theorem

$$\sum_{0}^{n}(-1)^i\alpha_i = 1 + (-1)^n + \sum_{1}^{n-1}(-1)^i(R_i - 1). \tag{5}$$

These relations hold good whether the manifold is one- or two-sided.

The numbers R_i are connected with the numbers P_i by a formula (§ 15) which involves the coefficients of torsion. They therefore do not supply us with any new invariants of a manifold. They seem to us, however, to deserve attention because they are connected in a fundamental way with the definition of a manifold and because of the generality and simplicity of the relations (3) and (4).

In §§ 17, 18, we show how a manifold may be described in a simple manner by means of a single matrix.

The Cell and the Complex.

3. An *n-dimensional simplex* will be defined as that one among the regions into which n-space is subdivided by $n + 1$ linearly independent $(n - 1)$-spaces which does not contain a point at infinity. Thus, the interior of a triangle in a plane is a two-dimensional simplex, and the linear segment joining two points is a one-dimensional simplex.

* These equations are generalizations of those employed by O. Veblen in pp. 86–94 of this volume of the *Annals*. The connection of the operation of reducing modulo 2 with the definition of a manifold seems first to have been noted by H. Tietze, Monatshefte für Mathematik und Physik, vol. 19 (1908), p. 49. Tietze defines a set of constants Q_i analogous to the R's but which do not satisfy either a duality or an Euler theorem. They are intended for a different purpose from ours.

Now let $[P]$ be a set of objects (e. g., points or lines,—we shall always refer to these objects as points) such that there exists a one-to-one reciprocal correspondence between $[P]$ and the points interior to and on the boundary of an n-dimensional simplex. Then the points of $[P]$ which correspond to the interior of the simplex are said to constitute an *n-dimensional cell* E_n, and those which correspond to the boundary of the simplex are said to constitute the *boundary of the cell.*

We define the order relations among the points of E_n and its boundary in terms of the order relations among the images of these points on the simplex and its boundary. Moreover, when we say that a cell a is on the boundary of another cell b, we always mean to imply that the correspondence between the points of a and those of the simplex defining a is continuous with respect to the order relations among the points of b and its boundary.

4. Now consider a set C_n of cells consisting of

α_0	0-cells (points)	x_1^0,	x_2^0,	x_3^0,	$\cdots$,	$x_{\alpha_0}^0$
α_1	1-cells (arcs of curve)	x_1^1,	x_2^1,	x_3^1,	$\cdots$,	$x_{\alpha_1}^1$
α_2	2-cells	x_1^2,	x_2^2,	x_3^2,	$\cdots$,	$x_{\alpha_2}^2$
. . .	. . .	.	.	.	. . .	.
α_i	i-cells	x_1^i,	x_2^i,	x_3^i,	$\cdots$,	$x_{\alpha_i}^i$
. . .	. . .	.	.	.	. . .	.
α_n	n-cells	x_1^n,	x_2^n,	x_3^n,	$\cdots$,	$x_{\alpha_n}^n$

The set C_n will be called a *complex* provided the following conditions are satisfied:

(1) The boundary of every i-cell ($i > 0$) is made up entirely of cells x_k^j of dimensionalities less than i.

(2) Every i-cell ($i < n$) is on the boundary of some $(i + 1)$-cell x_k^{i+1}.

For example, the figure which is obtained when a projective 3-space is divided up into cells by means of a tetrahedron is a complex of a simple type. It is made up of four points, twelve 1-cells, sixteen 2-cells, and eight 3-cells. Thus, $\alpha_0 = 4$, $\alpha_1 = 12$, $\alpha_2 = 16$, and $\alpha_3 = 8$.

Description of a Complex By Means of Matrices.

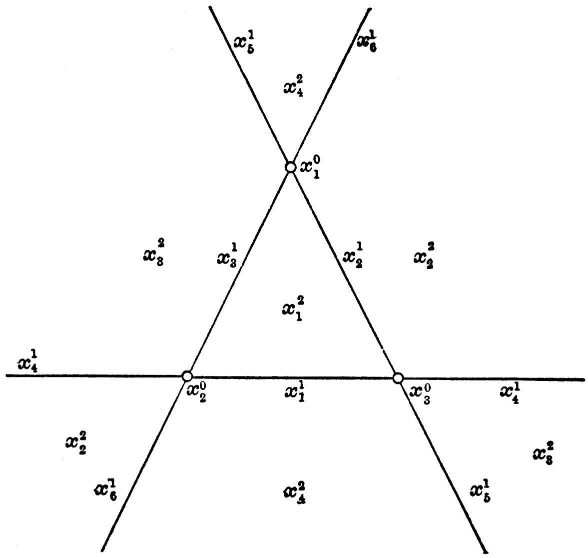
5. The structure of any n-dimensional complex C_n may be completely described by means of n suitably chosen tables or matrices $X_{0,1}$, $X_{1,2}$, $X_{2,3}$, $\cdots$, $X_{n-1,n}$. The matrix $X_{i-1,i}$ is an array of α_i columns and α_{i-1} rows, each column being associated with a distinct i-cell, and each row being associated with a distinct $(i - 1)$-cell of the complex C_n. In the jth row and kth column, there appears a number η_{jk}^i which is equal to

unity if the cells which correspond to the jth row and kth column respectively are adjacent, but otherwise equal to zero.*

6. On page 87 of this volume of the *Annals* are given the matrices which describe the surface of a tetrahedron. We give below the matrices for the real projective plane when subdivided into triangles by means of three straight lines.

	x_1^1	x_2^1	x_3^1	x_4^1	x_5^1	x_6^1
x_1^0	0	1	1	0	1	1
x_2^0	1	0	1	1	0	1
x_3^0	1	1	0	1	1	0

	x_1^2	x_2^2	x_3^2	x_4^2
x_1^1	1	0	0	1
x_2^1	1	1	0	0
x_3^1	1	0	1	0
x_4^1	0	1	1	0
x_5^1	0	0	1	1
x_6^1	0	1	0	1



To describe the complex which we mentioned at the end of § 4 will require three matrices. The first will have four rows and twelve columns; the second, twelve rows and sixteen columns; and the third, sixteen rows and eight columns.

7. A complex C_n is said to be *closed* if every one of its $(n-1)$-cells is upon the boundary of an even number of n-cells; otherwise, it is said to be *open* or *bounded*, and its boundary is said to consist of those $(n-1)$-cells (together with their boundaries) upon which an odd number of n-cells abut.

If a closed complex C_n contains no closed n-dimensional sub-complex, it is called an *n-dimensional circuit*. A 1-circuit, according to this definition, is a simple curve composed of a chain of arcs.

8. Let us remark in passing that Poincaré sometimes permits the boundary of an i-cell to touch itself along one or more cells of lower dimensionalities. When a complex contains cells of this more general type, it cannot always be completely characterized by means of the matrices $X_{0,1}, X_{1,2}, X_{2,3}, \cdots, X_{n-1,n}$.† The description may be made somewhat

* The relation between these matrices and the matrices of Poincaré (Lond. Math. Soc. Proc., vol. 32, p. 280) is that $\eta_{jk}^i = |\epsilon_{kj}^i|$. Clearly, the sign of η_{jk}^i is not relevant to the *definition* of a manifold.

† The same may be said about the matrices of Poincaré.

more precise if, in constructing the matrices, we make use of integers other than 1 whenever we wish to indicate that an i-cell appears more than once upon the boundary of an $(i+1)$-cell, but a simple example will show that ambiguity may arise even in this case.

Take two triangles ABC and $A'B'C'$ and deform them in such a way that the six vertices $ABCA'B'C'$ coincide. A complex C_2 may then be obtained if the arcs AB, BC, and CA be deformed into coincidence with the arcs $A'B'$, $B'C'$, and $C'A'$ respectively. If however we deform the arcs AB, BC, and CA into coincidence with $B'C'$, $A'B'$, and $C'A'$ respectively, we obtain a different complex $\bar{C}_2$ which determines the same matrices as C_2. We may verify that C_2 and $\bar{C}_2$ are distinct by observing that the neighborhood of the vertex of C_2 has three sheets, whereas the neighborhood of the vertex of $\bar{C}_2$ has two.

Manifolds.

9. The totality of points in the various cells of a complex C_n constitutes an ordered set $[P]$. This set will be called a *manifold* M_n if and only if it has the following three properties.

1. That every point P is interior to some n-cell of $[P]$.* (This condition is obviously satisfied by all the points interior to the n-cells of C_n, but would not in general be satisfied by the points on their boundaries.)

2. That if two n-cells E_n^1 and E_n^2 of $[P]$ have a point in common, there exists an n-cell contained within each of the cells E_n^1 and E_n^2.

3. That if P and P' be any two points of C_n, there always exists a chain of overlapping n-cells connecting an n-cell about P to an n-cell about P'.

In reading these conditions, it should be remembered that a cell, by definition, contains no point of its own boundary. The conditions imply that every $(n-1)$-cell x_i^{n-1} of a manifold is on two and only two n-cells, that every x_i^{n-2} is on a set of x_j^n's and x_k^{n-1}'s which are related like a set of points and arcs making up a circle, that every x_i^{n-3} is on a set of x_j^n's, x_k^{n-1}'s, and x_l^{n-2}'s which are related like the points, arcs, and spherical regions forming the surface of a sphere, and so on. Among the possibilities which are excluded are surfaces such as the complete cone, because of the singularity at the apex, and complexes such as the one obtained by partitioning the surface of an anchor ring into cells, selecting a point P not on the 3-space of the anchor ring and erecting upon each cell of the anchor ring as base a pyramid with P as apex.†

* When we say that a is a cell of $[P]$, we always imply that the correspondence between a and the simplex which defines a is continuous with respect to the order relations among the points of $[P]$ (cf. § 3).

† Encyklopädie der Mathematischen Wissenschaften, III, A, B, 3 (Dehn and Heegard), p. 183.

10. It is evident that many complexes give rise to the same or equivalent manifolds, for every manifold may be partitioned into cells in an indefinitely large number of ways. Hence the problem arises to determine certain invariants which characterize the matrices of all complexes associated with the same manifold.

The problem has been completely solved for the case $n = 2$, for it has been found that a manifold M_2 is completely determined when we know whether it is one or two-sided* and what is the value of the expression $\alpha_0 - \alpha_1 + \alpha_2$. For the cases where n is greater than 2, however, no complete set of invariants has yet been discovered.

The Generalized Euler Formula and the Constants R.

11. Let us regard the symbols x_i^j associated with the various rows and columns of the matrices as variables which may take on the values 0 and 1, and which are combined by reducing modulo 2. Then corresponding to every jth row of the matrix $X_{i-1,\,i}$ we shall write an equation of the form

$$(X_{i-1,\,i}) \qquad \sum_{k=1}^{\alpha_i} \eta_{jk}^i x_k^i = 0 \qquad j = 1, 2, 3, \cdots \alpha_{i-1},\dagger$$

where we recall that η_{jk}^i is the number which appears in the jth row and kth column of the matrix $X_{i-1,\,i}$, and where x_1^i, x_2^i, x_3^i, $\cdots$, $x_{\alpha_i}^i$ are the variables which correspond to the various columns of $X_{i-1,\,i}$. In this manner, a modular equation is associated with every $(i-1)$-cell of C_n, the right-hand member of which is zero and the left-hand member of which consists of the sum of the symbols x_k^i corresponding to the i-cells which abut upon the $(i-1)$-cell in question. (The coefficients of the remaining i-cells are all zero.)

Now, a solution of the system of equations $(X_{i-1,\,i})$ marks every i-cell of C_n with a number 0 or 1. Moreover, since no equation of the set is satisfied unless an even number or none of its variables take on the value 1, we see that among the i-cells which abut upon any given $(i-1)$-cell of the complex, an even number or none will always be marked with a 1. In other words, every solution of $(X_{i-1,\,i})$ defines an i-circuit or a system of i-circuits; and, conversely, every i-circuit or system of i-circuits defines a solution of $(X_{i-1,\,i})$. In particular, every column of the matrix $X_{i,\,i+1}$ defines a solution of the system $(X_{i-1,\,i})$, for it marks with a 1 all of the i-cells upon the boundary of one of the $(i+1)$-cells. The converse is obviously not true, since there are in general many circuits which are not the boundaries of cells of C_n.

*The distinction between one- and two-sidedness will be made in § 13.

† These equations have been explained in a simple case on pp. 86–94 of this volume of the *Annals*.

If two solutions s_1 and s_2 be added, their sum is a solution s_3 which is linearly dependent upon s_1 and s_2. Geometrically, s_3 will be represented by the set of circuits which one obtains by superimposing the circuits s_1 upon the circuits s_2 and leaving off the i-cells which are common to the two systems. Thus, if the boundaries of a set of $(i+1)$-cells be added according to the above rule, the resulting system of i-circuits will be the boundary of one or more $(i+1)$-complexes (§ 7) consisting of the $(i+1)$-cells in question. Conversely, the boundaries of one or more open $(i+1)$-complexes can always be expressed as a sum of the boundaries of the $(i+1)$-cells which make up the $(i+1)$-complexes.

12. We shall denote the maximum number of linearly independent solutions of $(X_{i-1,\,i})$ by $\sigma_{i-1,\,i}$. Then the total number of solutions will be $2^{\sigma_{i-1,\,i}}$, since every solution of $(X_{i-1,\,i})$ is of the form

$$\sum_{j=1}^{\sigma_{i-1,\,i}} \lambda_j c_j \qquad (\lambda = 0,\, 1)$$

where $c_1, c_2, c_3, \cdots, c_{\sigma_{i-1,\,i}}$ are a complete set of linearly independent solutions.

Now by a well-known theorem which is true for modular as well as for ordinary linear equations,

$$\sigma_{i-1,\,i} = \alpha_i - \rho_{i-1,\,i}$$

where $\rho_{i-1,\,i}$ is the order of the non-vanishing determinant of highest order in the matrix $X_{i-1,\,i}$ (i. e., the rank of $X_{i-1,\,i}$), and α_i is defined as in § 4. The count may also be made in another way. For we have seen that every column of the matrix $X_{i,\,i+1}$ yields a solution of the equations $(X_{i-1,\,i})$, and since the rank of $X_{i,\,i+1}$ is $\rho_{i,\,i+1}$, the number of linearly independent solutions of this type must also be $\rho_{i,\,i+1}$. Combinations of these solutions give the boundaries of open $(i+1)$-complexes, as we have seen. In general, there will also exist non-bounding i-circuits which cannot be expressed linearly in terms of the boundaries of cells. Let $(R_i - 1)$ *be the number of i-circuits which must be added to the bounding i-circuits before we can obtain a complete set of linearly independent solutions.* Then

$$\sigma_{i-1,\,i} = \rho_{i,\,i+1} + (R_i - 1),$$

and, equating the two values for $\sigma_{i-1,\,i}$,

$$\alpha_i - \rho_{i-1,\,i} = \rho_{i,\,i+1} + (R_i - 1).$$

The value of $\rho_{0,\,1}$ may be determined directly by special considerations. For the number 1 will appear just twice in every column of the matrix $X_{0,\,1}$ since every arc has two end points. Consequently, if we add together all the equations $(X_{0,\,1})$, their sum must be identically zero, modulo 2;

or in other words, the last column must be equal to the sum of the first $\alpha_0 - 1$ columns, and $\rho_{0,1}$ cannot be greater than $\alpha_0 - 1$. Nor is $\rho_{0,1}$ less than $\alpha_0 - 1$. For if it were, the sum of some subset $(X'_{0,1})$ of the equations would vanish identically, and we could subdivide the vertices of C_n into two classes, the first being made up of the vertices corresponding to the equations $(X'_{0,1})$ and the second, of the remaining ones. And there would be no arc x^1_i joining a vertex of the first class to one of the second class, otherwise the symbol x^1_i would appear once and only once in the equations $(X'_{0,1})$, and the sum of these equations could not vanish identically. Thus, if $\rho_{0,1}$ were less than $\alpha_0 - 1$, the manifold would not be connected, contrary to definition.

Thus, we have that

$$\alpha_0 - 1 = \rho_{0,1}.$$

Arguing in a similar way, we show that the rank $\rho_{n-1,n}$ of the matrix $X_{n-1,n}$ is $\alpha_n - 1$. For in every row, there are two 1's since two and only two n-cells abut upon the same $(n-1)$-cell. Consequently, the sum of all the columns vanishes identically, proving that $\rho_{n-1,n}$ is not greater than $\alpha_n - 1$. But the sum of a smaller number of columns than $\alpha_n - 1$ cannot vanish identically, otherwise these columns would define a set of n-cells having no $(n-1)$-cell in common with the remaining ones, which would again mean that C_n was not connected.

Hence,

$$\alpha_n - \rho_{n-1,n} = 1.$$

In all, we have the following relations:

$$
\begin{aligned}
\alpha_0 - 1 &= \rho_{0,1}, \\
\alpha_1 - \rho_{0,1} &= \rho_{1,2} + (R_1 - 1), \\
\alpha_2 - \rho_{1,2} &= \rho_{2,3} + (R_2 - 1), \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
\alpha_{n-2} - \rho_{n-3,n-2} &= \rho_{n-2,n-3} + (R_{n-2} - 1), \\
\alpha_{n-1} - \rho_{n-2,n-1} &= \rho_{n-1,n} + (R_{n-1} - 1), \\
\alpha_n - \rho_{n-1,n} &= 1.
\end{aligned} \tag{6}
$$

Multiplying these equations alternately by $+1$ and -1 and adding, the ρ's disappear and we obtain the relation

$$\sum_0^n (-1)^i \alpha_i = 1 + (-1)^n + \sum_1^{n-1} (-1)^i (R_i - 1), \tag{5}$$

which is the same as (5), § 1.

The Notion of Sense.

13. Let us make the convention that every arc of the complex C_n shall be positively related to one of its end-points and negatively related to the other. We can then assign to every arc one of two "senses" according as we say that it is positively related to one or the other of its end-points.* The sensed arc x_i^1 will be denoted by the symbol $\pm a_i^1$, the positive sign being associated with one determination of sense, the negative sign with the other.

The way in which we choose to assign the senses to the various arcs of C_n may be indicated by modifying the matrix $X_{0,\,1}$. Leaving the symbol 0 unchanged wherever it occurs, we shall replace the symbol 1 by the symbol -1 whenever the arc and point to which it corresponds are negatively related. If the arc and point are positively related, we shall leave the 1 unchanged.

In every column of the new matrix $A_{0,\,1}$ which we thus obtain, the numbers 1 and -1 each appear once. The sign of one of the 1's in each of the columns may be selected arbitrarily, after which, the sign of the other will be determined. Every choice corresponds to a different way of giving senses to the arcs.

We shall say that two arcs which abut upon the same point P have the same sense if one is positively and the other negatively related to P, and that they have opposite senses if they are similarly related to P. Then it is easy to see that from any one-dimensional circuit, there may be derived two sensed circuits, where in a sensed circuit, every arc has the same sense as each of the two arcs upon which it abuts. For as soon as we assign a sense to one of the arcs, the senses of all the other arcs are uniquely determined. Either of two senses may thus be assigned to the boundary of a two-cell.

To assign sense to a 2-cell x_i^2, we may make the convention that x_i^2 is positively related to its boundary c taken in one sense and negatively related to its boundary $-c$ taken in the opposite sense. x_i^2 will then be said to be positively related to the cells of c and negatively related to the cells of $-c$. To give the opposite sense to x_i^2, we say that it shall be positively related to $-c$.

If we replace 1 by -1 in the matrix $X_{1,\,2}$ whenever we wish to indicate that a 2-cell and an adjacent 1-cell are negatively related, we shall obtain a matrix $A_{1,\,2}$ in which the senses of the 2-cell are indicated. If the sense of any 1-cell be changed, so must be the signs of all the 1's in the corresponding row of $A_{1,\,2}$, while if the sign of a 2-cell be changed, so must be the signs of the 1's in the corresponding column.

* The connection of this statement with the intuitive notion of sense is obvious.

Now if we make the convention that two adjacent 2-cells have the same or opposite senses according as they are oppositely or similarly related to a common arc, we cannot conclude as we could in the linear case that two sensed 2-circuits can always be derived from an unsensed circuit c_2. When we can, c_2 is said to be *two-sided*; when we cannot, c_2 is said to be *one-sided.* The boundary of a 3-cell is two-sided; the projective plane, one-sided, as may be verified by an examination of the matrices in § 6.

Finally, we may assign senses to the 3-cells, 4-cells, $\cdots$, n-cells of C_n just as we did to the 2-cells, and can obtain a set of matrices $A_{0,\,1}$, $A_{1,\,2}$, $A_{2,\,3}$, $A_{3,\,4}$, $\cdots$, $A_{n-1,\,n}$ which not only define the complex C_n but also the sense of every cell of C_n (except the 0-cells or points, which are not assigned senses).

14. Now, let us consider a set of ordinary *non-modular* linear equations

$$(A_{i-1,\,i}) \qquad \sum_{k=1}^{\alpha_i} \epsilon_{jk}{}^i x_k{}^i = 0 \qquad j = 1, 2, 3, \cdots, \alpha_{i-1},$$

which can be derived from the matrix $A_{i-1,\,i}$ just as the equations $(X_{i-1,\,i})$ were derived from the matrix $X_{i-1,\,i}$.

Then every solution of the system $(A_{i-1,\,i})$ in integers will correspond to a sensed i-circuit or system of i-circuits, provided that we make the convention that in a circuit the same cell may appear and be counted more than once. Moreover, every solution of $(A_{i-1,\,i})$ is linearly dependent upon a set of integer solutions, and hence we may say that the number of linearly independent sensed (and therefore two-sided) circuits is equal to the number of linearly independent solutions of $(A_{i-1,\,i})$.

The operation of combining sensed circuits which corresponds to adding solutions of the above equations is not the same as the operation of combining unsensed ones which we previously considered. For if two sensed circuits having a common cell are added, the coincident cells do not annul one another unless they have opposite senses.

Now, by the theorem on linear equations which we used when working with the modular equations, if $\mu_{i-1,\,i}$ be the number of linearly independent solutions of the system $(A_{i-1,\,i})$, then

$$\mu_{i-1,\,i} = \alpha_i - \nu_{i-1,\,i}$$

where $\nu_{i-1,\,i}$ is the rank of the matrix $A_{i-1,\,i}$. Furthermore, since the boundary of every $(i + 1)$-cell taken in a definite sense is an i-circuit, the columns of the matrix $A_{i,\,i+1}$ must define solutions of the system $(A_{i-1,\,i})$. And since the rank of $A_{i,\,i+1}$ is $\nu_{i,\,i+1}$, the number of linearly independent solutions of this sort is $\nu_{i,\,i+1}$. *Let $(P_i - 1)$ be the number of sensed i-circuits*

which must be added to the bounding i-circuits before we can obtain a complete set of linearly independent solutions. Then the relation

$$\mu_{i-1,\, i} = \nu_{i,\, i+1} + (P_i - 1)$$

must also hold. Equating the two values of $\mu_{i-1,\, i}$ we have

$$\alpha_i - \nu_{i-1,\, i} = \nu_{i,\, i+1} + (P_i - 1).$$

The values of $\nu_{0,1}$ and $\nu_{n-1,n}$ may be calculated directly by special considerations analogous to those used in determining $\rho_{0,1}$ and $\rho_{n-1,n}$ in connection with the modular equations. The sum of all the equations $(A_{0,\,1})$ vanishes identically, whereas if a linear combination of less than all vanished, the vertices could be divided into sets corresponding to disconnected portions of C_n. Hence,

$$\alpha_0 - 1 = \nu_{0,\,1}.$$

Furthermore, no linear combination of less than all the columns of the last matrix $A_{n-1,\, n}$ can vanish; otherwise, the n-cells defined by such columns would not be connected with the rest of the manifold by any $(n - 1)$-cell. We cannot conclude, however, that there exists a linear combination of all of the columns which vanishes identically. Two cases arise according as C_n is one- or two-sided. If C_n is two-sided, then senses may be so assigned to the various n-cells that every $(n - 1)$-cell is oppositely related to the two n-cells which abut upon it. In other words, the sum of the columns, with the proper sign given to each, vanishes, and we have

$$\nu_{n-1,\, n} = \alpha_n - 1.$$

For one-sided complexes no such relation exists and

$$\nu_{n-1,\, n} = \alpha_n.$$

Thus, we have a criterion for one- and two-sidedness.

Altogether, we have the relations

$$
\begin{aligned}
\alpha_0 - 1 &= \nu_{0,\,1}, \\
\alpha_1 - \nu_{0,\,1} &= \nu_{1,\,2} + (P_1 - 1), \\
\alpha_2 - \nu_{1,\,2} &= \nu_{2,\,3} + (P_2 - 1), \qquad (7) \\
&\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots \\
\alpha_{n-1} - \nu_{n-2,\,n-1} &= \nu_{n-1,\,n} + (P_{n-1} - 1),
\end{aligned}
$$

$$
\left.
\begin{aligned}
\alpha_n - \nu_{n-1,\,n} &= 1 \\
\text{or} \qquad \alpha_n - \nu_{n-1,\,n} &= 0
\end{aligned}
\right\}.
$$

From which we derive the formulas

(2) $$\sum_0^n (-1)^i \alpha_i = 1 + (-1)^n + \sum_1^{n-1} (-1)^i (P_i - 1)$$
and
(3) $$\sum_0^n (-1)^i \alpha_i = 1 + \sum_1^{n-1} (-1)^i (P_i - 1)$$

for two- and one-sided manifolds respectively. These are the Euler-Poincaré equations, and the numbers

$$P_1, \quad P_2, \quad P_3, \quad \cdots, \quad P_{n-1}$$

are the Betti numbers of Poincaré.*

Relation between the Numbers R and P.

15. Let $X_{0,1}, X_{1,2}, \cdots, X_{n-1,n}$ be a set of matrices of the first type (§ 5) which describe a manifold M_n, and let $A_{0,1}, A_{1,2}, \cdots, A_{n-1,n}$ be the corresponding set of the second type (§ 13). Then each of the matrices $A_{i,i+1}$ may be reduced by means of elementary transformations "without division"† to a matrix $\bar{A}_{i,i+1}$ where all the elements of $\bar{A}_{i,i+1}$ which are not on the main diagonal are zero and the elements of the main diagonal are the invariant factors of $A_{i,i+1}$. The rank of $A_{i,i+1}$ is the same as the number of non-vanishing elements of $\bar{A}_{i,i+1}$. Whenever a number other than 0 or 1 appears in one of the reduced matrices, the absolute value of that number is said to be a *coefficient of torsion* of the manifold M_n. The coefficients of torsion are invariants of M_n.‡

Now the rank of the matrix $X_{i,i+1}$ *modulo 2* is the same as that of the matrix $A_{i,i+1}$ *modulo 2*, for the elements of $X_{i,i+1}$ differ at most in sign from the corresponding elements of $A_{i,i+1}$. Hence, the rank of $X_{i,i+1}$ is equal to the rank of $\bar{A}_{i,i+1}$ after the elements along the diagonal of $\bar{A}_{i,i+1}$ have been reduced modulo 2. But the effect of this reduction is to replace the even coefficients of torsion by 0's and the odd ones by 1's. Hence,

(8) $$\nu_{i,i+1} - \rho_{i,i+1} = t_{i,i+1}$$

where $\nu_{i,i+1}$ is the rank of $A_{i,i+1}$; $\rho_{i,i+1}$, the rank of $X_{i,i+1}$ modulo 2; and $t_{i,i+1}$ the number of even coefficients of torsion of $A_{i,i+1}$.

But in §§ 12 and 14 respectively, we derived the two relations

(6) $$\alpha_i - \rho_{i-1,i} = \rho_{i,i+1} + (R_i - 1),$$
and
(7) $$\alpha_i - \nu_{i-1,i} = \nu_{i,i+1} + (P_i - 1),$$
whence we have

* Palermo Rendiconti, vol. 13 (1899), p. 286 and p. 301.

† Poincaré, Lond. Math. Soc. Proc., vol. 32 (1900), p. 286.

‡ Loc. cit., p. 286 et seq.

$$R_i - P_i = (\nu_{i-1,\,i} - \rho_{i-1,\,i}) + (\nu_{i,\,i+1} - \rho_{i,\,i+1})$$

or

(9) $$R_i = P_i + t_{i-1,\,i} + t_{i,\,i+1},$$

which gives the relation between the constants R and the constants P found by Poincaré. In comparing equations (2) and (3) with (5) by means of (9), it must be remembered that $t_{0,\,1} = 0$ and $t_{n-1,\,n}$ is 0 or 1 according as the manifold is two- or one-sided.

16. Since Poincaré has shown that the Betti numbers and the coefficients of torsion are invariants of a manifold,* the invariance of the numbers R follows at once from Equation (9). And since he has also shown that for two-sided manifolds

$$P_i = P_{n-i}$$

and

$$t_{i-1,\,i} = t_{n-i,\,n-i+1},$$

it also follows that for two-sided manifolds,

$$R_i = R_{n-i}.$$

That this last relation also holds for one-sided manifolds will follow as a result of the discussion in § 18.

Regular Subdivision.

17. We have seen that every manifold M_n may be described by means of the n matrices associated with any complex C_n which defines M_n. We shall now prove that if the cells of C_n be suitably subdivided, a complex $\bar{C}_n$ may always be obtained which is of such a simple type that it is completely characterized by a single matrix. A subdivision of the required sort is given below:

First, we introduce a new vertex upon each arc of C_n, thereby subdividing it into two arcs, *no two of which are bounded by the same pair of end-points.* For convenience, we shall call these new arcs one-dimensional "pyramids." The apexes of these "pyramids" will be the new vertices and the bases will be the old vertices of C_n.

Secondly, we introduce a new vertex upon every 2-cell of C_n and join each of these vertices by arcs to the vertices upon the boundary of the cell in which it lies. The 2-cells are thus decomposed into triangular regions or two-dimensional "pyramids" which are bounded by arcs or one-dimensional pyramids. The new vertices are the apexes of the new pyramids, the old points and arcs, their bases.

* It must be remarked here that Poincaré assumes in proving the invariance of the P's that every cell of C_n is made up of a finite number of analytic pieces.

Thirdly, by a similar process, we subdivide the 3-cells into three-dimensional pyramids bounded by one- and two-dimensional ones, and so on.*

After we have subdivided all of the cells of C_n in the above manner, we shall have a complex $\bar{C}_n$ of a very simple type. For, every two-cell of $\bar{C}_n$ is bounded by three arcs; every 3-cell, by four triangular faces; every 4-cell, by five tetrahedral faces; and so on. In other words, not only will it be possible to map every k-dimensional cell E_k of $\bar{C}_n$ along with its boundary upon the interior and boundary of a k-dimensional simplex S_k, but the mapping may be made in such a way that the image of every 0-cell on the boundary of E_k is a vertex of S_k; the image of every 1-cell on the boundary of E_k, an edge of S_k; the image of every 2-cell on the boundary of E_k, a face of S_k; and so on.

Let us also observe that if E_i^1 and E_i^2 be any two i-cells of $\bar{C}_n$, then the vertices of $\bar{C}_n$ which lie upon the boundary of E_i^1 cannot all lie upon the boundary of E_i^2. If E_i^1 and E_i^2 are arcs, the truth of this statement is obvious from the construction of $\bar{C}_n$; in the higher cases, the proof may be made as follows:

Suppose the same $i + 1$ vertices appeared upon the boundaries of both E_i^1 and E_i^2. Then that one of the $i + 1$ vertices which was the last to be introduced during the subdivision of C_n would necessarily be the apex of both the "pyramids" E_i^1 and E_i^2, while the remaining i vertices would lie upon the boundaries of the bases of E_i^1 and E_i^2. Calling these bases E_{i-1}^1 and E_{i-1}^2 respectively, we could apply the argument just made to these two new cells and thereby show that there existed two $(i - 2)$-cells E_{i-2}^1 and E_{i-2}^2 upon whose boundaries the same $i - 1$ vertices of $\bar{C}_n$ appeared. And after $i - 1$ steps of this sort, we should be led to the conclusion that two arcs E_1^1 and E_1^2 of $\bar{C}_n$ were bounded by the same pair of points.

18. Now suppose we consider a simplex S having the same number of vertices as the complex $\bar{C}_n$. Then if we denote the vertices of $\bar{C}_n$ by

$$x_1^0, \quad x_2^0, \quad x_3^0, \quad \cdots, \quad x_p^0$$

and the vertices of S by

$$V_1, \quad V_2, \quad V_3, \quad \cdots, \quad V_p,$$

it is clear that to every vertex x_i^0 of $\bar{C}_n$ may be made to correspond the vertex V_i of S. And in a like manner, to every one-cell $x_i^0x_j^0$ of $\bar{C}_n$ may be made to correspond the one-cell V_iV_j of S; to every 2-cell $x_i^0x_j^0x_k^0$, the 2-cell $V_iV_jV_k$; and so on. Moreover, no two cells of $\bar{C}_n$ are made to correspond to the same cell of S, otherwise the boundaries of both would include the same vertices of C_n, which we proved to be impossible.

* Cf. Poincaré, Palermo Rendiconti, vol. 13, p. 314.

We thus have the theorem that *every complex may be subdivided into a complex $\bar{C}_n$ which is equivalent to a complex D_n the elements of which are a subset of the elements upon the boundary of a simplex of $p - 1$ dimensions.*

Now a complex of the type $\bar{C}_n$ may be described by means of a single matrix X the rows and columns of which are in one-to-one correspondence with the vertices and n-cells of $\bar{C}_n$ respectively. We put 1 or 0 in the ith row and jth column according as the point x_i^0 is or is not on the n-cell x_j^n. To prove that the matrix X describes the complex $\bar{C}_n$ we need only observe that every k-cell of $\bar{C}_n$ is determined by $k + 1$ vertices which lie upon the boundary of the same n-cell of $\bar{C}_n$; while, conversely, every set of $k + 1$ vertices which lie upon the boundary of the same n-cell determines one and only one k-cell. Consequently, $k + 1$ given vertices of $\bar{C}_n$ will determine a k-cell if and only if there exists a column in the matrix X which contains a 1 in each of the rows corresponding to the vertices in question.

Dual Complexes.

19. If C_n be a complex which defines a manifold M_n, there always exists a complex C'_n which is dual to C_n and which also defines M_n; where two complexes C_n and C'_n are said to be *dual* if the points, arcs, $\cdots$, k-cells, $\cdots$ of the one are in one-to-one correspondence with the n-cells, $(n - 1)$-cells, $\cdots$, $(n - k)$-cells, $\cdots$ respectively of the other, the correspondence being such that two cells of C'_n abut upon one another if and only if the corresponding cells of C_n do.

A simple way of showing the existence* of C'_n is to observe that if a regular subdivision be applied to the complex C_n, a complex $\bar{C}_n$ is thereby obtained which contains the required complex C'_n as a sub-complex. C'_n may in fact be derived from $\bar{C}_n$ by applying what we may term the inverse of a regular subdivision.

For, by the definition of a manifold, the n-cells of $\bar{C}_n$ which abut upon one of the vertices V_0 of the original complex C_n constitute, with the cells which separate them from one another, an n-cell E_n. The inverse of a regular subdivision applied to these cells will amount to replacing the vertex V_0 and the adjacent cells by the single cell E_n. By making a similar transformation about every vertex of $\bar{C}_n$ which also belongs to the original complex C_n, we shall obtain a set of n-cells which are in one-to-one correspondence with the vertices of C_n and which will serve as the n-cells of the dual complex C'_n.

Upon the boundaries of the n-cells E_n will appear the points V_1 which we introduced in subdividing the arcs of C_n. We can apply the inverse of a regular subdivision to the remaining cells of dimensionality $n - 1$ or less which

* Poincaré, Palermo Rendiconti, vol. 13, p. 314.

cluster about each of these points V_1, thereby obtaining new $(n-1)$-cells which are in one-to-one correspondence with the arcs of C_n and will serve as the $(n-1)$-cells of C'_n. After the $(i+1)$st step, there will remain upon the boundaries of the $(n-i)$-cells E_{n-i} the points V_{i+1} which we introduced upon the $(i+1)$-cells of C_n when we passed from C_n to $\bar{C}_n$. The inverse of a regular subdivision applied to the cells of dimensionality $n-i-1$ and lower which cluster about the vertices V_{i+1} will give the cells E_{n-i-1} of C'_n. Thus we may construct a complex C'_n dual to C_n.

Let us denote by $X_{0,1}$, $X_{1,2}$, $X_{2,3}$, $\cdots$, $X_{n-1,n}$ the matrices of the complex C_n, and by $X'_{0,1}$, $X'_{1,2}$, $X'_{2,3}$, $\cdots$, $X'_{n-1,n}$ the matrices of C'_n. Then owing to the fact that there is a one-to-one correspondence between the i-cells of C_n and the $(n-i)$-cells of C'_n and that an i-cell of C_n abuts upon an $(i+1)$-cell if and only if the corresponding $(n-i)$-cell of C'_n abuts upon the corresponding $(n-i-1)$-cell, it follows that the matrix $X'_{n-i-1,n-i}$ is the same as the matrix $X_{i,i+1}$ with rows and columns interchanged, and hence that the two matrices have the same rank.

Now, from Equations (6), § 12, we have at once

$$\begin{aligned} R_i &= \alpha_i - \rho_{i-1,i} - \rho_{i,i+1}, \\ &= \alpha'_{n-i} - \rho'_{n-i,n-i+1} - \rho'_{n-i-1,n-i} = R'_{n-i}, \end{aligned}$$

where R_i and R'_i denote the space constants of C_n and C'_n respectively.

But owing to the invariance of the constants of a manifold, we have

$$R_{n-i} = R'_{n-i},$$

and hence

$$R_i = R_{n-i},$$

which proves the duality relation for both one- and two-sided manifolds.

The proof which we have given above is essentially the one which Poincaré uses in showing that

$$P_i = P_{n-i}$$

for two-sided manifolds. Having proved the invariance of the coefficients of torsion, it may also be used in proving the duality relation which the latter also satisfy for two-sided manifolds. Similar theorems do not hold for one-sided manifolds; for although a dual complex C'_n may always be found, as was shown above, we cannot in the one-sided case so assign the senses to the cells of C_n and C'_n that corresponding cells are similarly sensed.

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